DEFINITION OF THH

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Abstract. This document contains notes for the "Definition of THH" talk at MIT Talbot 2024. It discusses a few basic properties of THH for $\mathbb{E}_\infty$ ring spectra as well as a few example computations.

Contents

1. Definition

In the following document, we consider only $\mathbb{E}_\infty$ ring spectra. Hence we will use the following definition of THH:

Definition 1.1. Let $R \to A$ be a map of $\mathbb{E}_\infty$ ring spectra. Then we define

$$\mathrm{THH}(A/R) = A \otimes_{A \otimes_R A} A.$$

We will denote the THH of A relative to the sphere spectrum $\mathbb{S}$ by $\mathrm{THH}(A)$.

When R is an ordinary (discrete) commutative ring, we will use the notation $\mathrm{THH}(R) := \mathrm{THH}(HR)$, or more generally denote HR by R.

Additionally, for a map of discrete commutative rings $R \to A$, we will use the notation $\mathrm{HH}(A/R) := \mathrm{THH}(HA/HR)$, or $\mathrm{HH}(A) := \mathrm{THH}(HA/H\mathbb{Z})$. In this case, we have that

$$\mathrm{HH}(A/R) = A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}}_R A} A,$$

i.e. what is classically known as Shukla homology.

Furthermore, tensor (smash) products of spectra will be over $\mathbb{S}$ unless otherwise noted. Also, $\mathcal{S}$ will denote the ∞-category of spaces, and Sp will denote the ∞-category of spectra.

Lastly, everything is always assumed to be $\mathbb{E}_\infty$ or commutative in this document unless stated otherwise (in case I forget these adjectives somewhere).

1.1. **Universal property.** An alternative definition of THH for $\mathbb{E}_\infty$ ring spectra is by universal property: given an $\mathbb{E}_\infty$ ring A, $\mathrm{THH}(A)$ is the "free $\mathbb{E}_\infty$ A-algebra with S^1 action".

First recall that $\mathrm{CAlg}(\mathrm{Sp})$ is tensored over spaces; i.e. there is a functor $(-)^{\otimes(-)} : \mathrm{CAlg}(\mathrm{Sp}) \times \mathcal{S} \to \mathrm{CAlg}(\mathrm{Sp})$ which preserves colimits separately in each variable, and if $A \in \mathrm{CAlg}(\mathrm{Sp})$, then $A^{\otimes(-)}$ is left adjoint to $\mathrm{Map}_{\mathbb{E}_\infty}(A, -)$. We have that if $X \in \mathcal{S}$, then

$$A^{\otimes X} \simeq \operatorname*{colim}_X A$$

where the RHS is the colimit of the A-valued constant diagram

$$X \to \mathrm{pt} \to \mathrm{CAlg}(\mathrm{Sp}).$$

If we are viewing A as an $\mathbb{E}_\infty$ R-algebra and considering the tensor of $\mathbb{E}_\infty$ R-algebras over spaces, we will use the notation $A^{\otimes_R X}$. (Note: in many sources, $\mathrm{colim}_X A$ is denoted by $A \otimes X$ or $X \otimes A$, but the notation above is preferred by some, since e.g. it can be confused with $A \otimes \Sigma^\infty_+ X$.)

Then we have the following theorem:

Theorem 1.1 (McClure-Schwänzl-Vogt [MSV97])**.** Let A be an $\mathbb{E}_\infty$ ring. Then we have that

$$\mathrm{THH}(A) \simeq A^{\otimes S^1}.$$

In particular, the unit map $i : A \to \mathrm{THH}(A)$ (induced by $\mathrm{pt} \to S^1$) is initial among maps from A to $\mathbb{E}_\infty$ rings with S^1 action; i.e. it induces the following equivalence for every $\mathbb{E}_\infty$ ring B with S^1 action:

$$\mathrm{Map}^{S^1}_{\mathbb{E}_\infty}(\mathrm{THH}(A), B) \simeq \mathrm{Map}_{\mathbb{E}_\infty}(A, B).$$

Proof. S^1 is a pushout in $\mathcal{S}$, and we simply apply $A^{\otimes(-)}$:

$$\begin{tikzcd} * \amalg * \arrow[r] \arrow[d] & * \arrow[d] \arrow[dl, phantom, "\ulcorner"] & & A \otimes A \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ * \arrow[r] & S^1 & \xrightarrow{A^{\otimes(-)}} & A \arrow[r] & A^{\otimes S^1} \end{tikzcd}.$$

Here we used the fact that $A \otimes_{\mathbb{S}} A$ is the coproduct in $\mathrm{CAlg}(\mathrm{Sp})$ (meaning that this doesn't work for associative ring spectra, for example). Simultaneously, we have the pushout

$$\begin{tikzcd} A \otimes A \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ A \arrow[r] & A \otimes_{A \otimes A} A \end{tikzcd}$$

and so we have that $\mathrm{THH}(A) = A \otimes_{A \otimes A} A \simeq A^{\otimes S^1}$ as desired. The universal property follows from the equivalence

$$\mathrm{Map}^{S^1}_{\mathbb{E}_\infty}(\operatorname*{colim}_{S^1} A, B) \simeq \mathrm{Map}^{S^1}_{\mathcal{S}}(S^1, \mathrm{Map}_{\mathbb{E}_\infty}(A, B)) \simeq \mathrm{Map}_{\mathbb{E}_\infty}(A, B).$$

$\square$

2. Base change formulas

In this section, we describe a few basic properties of THH. Throughout this document, we will also the fact that THH is a symmetric monoidal functor $\mathrm{CAlg}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})$. In general, proofs of these facts can be found in [KN18] or in the Homotopy Theory Münster Youtube lectures, though the proofs we present may be a bit different.

Theorem 2.1. Let $R \to R'$ and $R \to A$ be maps of $\mathbb{E}_\infty$ rings. Then

$$\mathrm{THH}(A/R) \otimes_R R' \simeq \mathrm{THH}(A \otimes_R R'/R')$$

Proof. Let B be an $\mathbb{E}_\infty$ R'-algebra with S^1 action. Then we show that the LHS has the universal property of the RHS:

$$\begin{aligned}\mathrm{Map}_{R'}^{S^1}(\mathrm{THH}(A/R) \otimes_R R', B) &\simeq \mathrm{Map}_R^{S^1}(\mathrm{THH}(A/R), B)\\ &\simeq \mathrm{Map}_R(A, B)\\ &\simeq \mathrm{Map}_{R'}(A \otimes_R R', B)\\ &\simeq \mathrm{Map}_{R'}^{S^1}(\mathrm{THH}(A \otimes_R R'/R'), B)\end{aligned}$$

as desired, where we've applied the extension/restriction of scalars adjunction given by $- \otimes_R R'$ a couple times to obtain the above equivalences. $\square$

Theorem 2.2. Suppose that we have a commutative diagram of $\mathbb{E}_\infty$ ring maps

$$\begin{tikzcd} A & C \arrow[l] \arrow[r] & B \\ R \arrow[u] & T \arrow[l] \arrow[r] \arrow[u] & S. \arrow[u] \end{tikzcd}$$

Then we have that

$$\mathrm{THH}(A \otimes_C B/R \otimes_T S) \simeq \mathrm{THH}(A/R) \otimes_{\mathrm{THH}(C/T)} \mathrm{THH}(B/S).$$

We will not prove this formula explicitly; we will instead prove the following simpler case, since the style of proof used would be the same (i.e. by manipulations of universal properties).

Corollary 2.1. Let $R \to A$ be a map of $\mathbb{E}_\infty$ rings. Then

$$\mathrm{THH}(A/R) \simeq \mathrm{THH}(A) \otimes_{\mathrm{THH}(R)} R.$$

Proof. We want to show that the RHS satisfies the universal property of the LHS. Given an $\mathbb{E}_\infty$ R-algebra B with S^1-action, we will first show that

$$\mathrm{Map}_R^{S^1}(\mathrm{THH}(A) \otimes_{\mathrm{THH}(R)} R, B) \simeq \mathrm{Map}_{\mathrm{THH}(R)}^{S^1}(\mathrm{THH}(A), B).$$

In particular, there exists an $\mathbb{E}_\infty$ "augmentation" map $f : \mathrm{THH}(R) \to R$ given by

$$f : \mathrm{THH}(R) \simeq \operatorname*{colim}_{S^1} R \to \operatorname*{colim}_{\mathrm{pt}} R \simeq R.$$

This is a retraction onto the 0th layer, i.e. precomposed with the unit $i : R \to \mathrm{THH}(R)$, we obtain the identity on R

$$\mathrm{id}_R : R \xrightarrow{i} \mathrm{THH}(R) \xrightarrow{f} R,$$

and furthermore f is S^1-equivariant with respect to the trivial action on R.

Then using the restriction/extension of scalars adjunction along this map

$$- \otimes_{\mathrm{THH}(R)} R : \mathrm{CAlg}^{S^1}_{\mathrm{THH}(R)} \rightleftharpoons \mathrm{CAlg}^{S^1}_R : f^*.$$

we directly obtain the above equivalence (where we were a bit imprecise by writing B instead of f^*B).

Then we have the below equivalence between the triangles of S^1-equivariant maps and triangles of nonequivariant maps

$$\begin{tikzcd} \mathrm{THH}(R) \arrow[r, dashed] \arrow[dr, dashed] & \mathrm{THH}(A) \arrow[d, dashed] \\ & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} R \arrow[r] \arrow[dr] & A \arrow[d] \\ & B \end{tikzcd}$$

where the dashed maps are uniquely determined by the universal properties of $\mathrm{THH}(R)$ and $\mathrm{THH}(A)$:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{THH}(R) \arrow[r, dashed] \arrow[dr, dashed] & \mathrm{THH}(A) \arrow[d, dashed] \\ & B \end{tikzcd}$$

i.e. we have that

$$\mathrm{Map}^{S^1}_{\mathrm{THH}(R)}(\mathrm{THH}(A), B) \simeq \mathrm{Map}_R(A, B)$$

as desired. $\square$

3. Example computations

In this section, we discuss a few example computations.

3.1. HH **of polynomial algebras.** Let R be a discrete commutative ring. Let us now try to compute $\mathrm{HH}_*(R[x_1, \dots, x_n]/R)$. Since THH commutes with $\otimes$ and $R[x_1, \dots, x_n] \cong R[x]^{\otimes n}$, it suffices to compute $\mathrm{HH}_*(R[x]/R)$. We claim that

$$\mathrm{HH}_*(R[x]/R) \cong R[x] \otimes \Lambda(dx)$$

i.e. it's just $R[x]$ concentrated in degrees 0 and 1. To begin, let's consider

$$\mathrm{HH}(R[x]/R) = R[x] \otimes^{\mathbb{L}}_{R[x] \otimes^{\mathbb{L}}_R R[x]} R[x] \cong R[x] \otimes^{\mathbb{L}}_{R[a,b]} R[x]$$

where the $R[a,b]$ action on $R[x]$ is by $a,b \mapsto x$ and multiplication inside $R[x]$. Hence we seek to determine $\mathrm{Tor}_*^{R[a,b]}(R[x],R[x])$. We can resolve $R[x]$ as an $R[a,b]$-module as follows:

$$0 \to R[a,b] \xrightarrow{\cdot(a-b)} \to R[a,b] \to R[a,b]/(a-b) \cong R[x] \to 0$$

and thus we obtain the following complex

$$0 \to R[x] \otimes_{R[a,b]} R[a,b] \xrightarrow{\mathrm{id} \otimes \cdot (a-b)} R[x] \otimes_{R[a,b]} R[a,b] \to 0$$

but of course $\cdot(a-b)$ kills $R[x]$ as an $R[a,b]$ module, and so the above complex simply becomes

$$0 \to R[x] \xrightarrow{0} R[x] \to 0$$

as desired.

3.2. THH(MU)**.** There exists a formula for THH of Thom spectra which allows us to directly compute THH(MU). This formula can be found in [Blu08],[BCS10], and [Sch11]. Another more recent interpretation can be found in [RSV22]. We will list a few variants of this formula, which mostly differ by degrees of multiplicativity; we will not give any proofs of them.

Theorem 3.1. Let R be an $\mathbb{E}_\infty$ ring spectrum and X be a connected $\mathbb{E}_1$-space. Suppose we have a map $f : X \to BGL_1(R)$; then we have the following equivalence of R-modules

$$\mathrm{THH}(Mf/R) \simeq M(LBX \to BGL_1(R))$$

where the map on the RHS is determined by f. If f is an $\mathbb{E}_2$ map, the equivalence is as $\mathbb{E}_1$ R-algebras.

The next statement is useful for other classical computations, e.g. for THH of $\mathbb{F}_p$, $\mathbb{Z}/p^n$, and $\mathbb{Z}$, which can be shown to arise as Thom spectra of certain $\mathbb{E}_2$-maps.

Theorem 3.2. Let R be an $\mathbb{E}_\infty$ ring spectrum and X be a connected $\mathbb{E}_2$-space. Let $f : X \to BGL_1(R)$ be an $\mathbb{E}_2$ map, and assume that the $\mathbb{E}_2$ R-algebra structure on Mf extends to an $\mathbb{E}_3$ R-algebra structure. Then we have the following equivalence of $\mathbb{E}_1$ R-algebras

$$\mathrm{THH}(Mf/R) \simeq Mf \otimes \mathbb{S}[BX].$$

This last statement is the same as the previous, but where everything is $\mathbb{E}_\infty$:

Theorem 3.3. Let R be an $\mathbb{E}_\infty$ ring spectrum, and let G be an $\mathbb{E}_\infty$-group. Let $f : G \to BGL_1(R)$ be an $\mathbb{E}_\infty$ map. Then there is an equivalence of $\mathbb{E}_\infty$ R-algebras

$$\mathrm{THH}(Mf/R) \simeq Mf \otimes \mathbb{S}[BG].$$

To compute $\mathrm{THH}(MU)$, we can simply plug MU into this formula to obtain the result of the computation, as

$$MU \simeq \mathrm{colim}(BU \to BGL_1(\mathbb{S}) \to \mathrm{Sp})$$

and so we obtain the equivalence

$$\mathrm{THH}(MU) \simeq MU \otimes \mathbb{S}[BBU].$$

In particular, one can compute

$$\mathrm{THH}_*(MU) = MU_* \otimes \Lambda(x_1, x_2, \ldots)$$

where generators are in odd degrees.

3.3. $\mathrm{THH}(\mathbb{S}[t])$. Let $\mathbb{S}[t]$ and $\mathbb{S}[\mathbb{N}]$ denote $\Sigma^\infty_+\mathbb{N}$. For intuition purposes, let's begin by doing a simpler calculation, $\mathrm{THH}(\mathbb{S}[\mathbb{Z}])$.

In general, for G an $\mathbb{E}_1$-space, we have an equivalence of spectra

$$\mathrm{THH}(\mathbb{S}[G]) \simeq \Sigma^\infty_+ LBG,$$

where the circle action occurs by rotating the loops. One can see this equivalence via Theorem 3.1; first, the group ring $\mathbb{S}[G]$ arises as the Thom spectrum of the trivial map $G \to BGL_1(\mathbb{S})$:

$$\mathrm{colim}(G \to BGL_1(\mathbb{S}) \to \mathrm{Sp}) \simeq \underset{G}{\mathrm{colim}}(\mathbb{S}) \simeq \Sigma^\infty_+ G = \mathbb{S}[G].$$

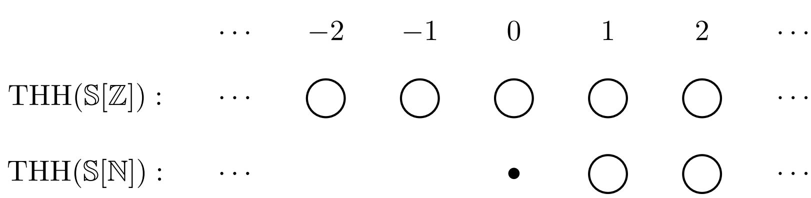
Then it turns out that that the induced map in the formula $LBG \to BGL_1(\mathbb{S})$ is also trivial, so $\mathrm{THH}(\mathbb{S}[G]) \simeq \Sigma^\infty_+ LBG$ as desired.

Thus, we have that

$$\mathrm{THH}(\mathbb{S}[\mathbb{Z}]) \simeq \Sigma^\infty_+ LS^1 \simeq \Sigma^\infty_+ \mathrm{Map}_{\mathcal{S}}(S^1, S^1) \simeq \Sigma^\infty_+ (S^1 \times \mathbb{Z})$$

since $\mathrm{Map}_{\mathcal{S}}(S^1, S^1)$ is determined by the choice of basepoint and the number of times you loop around (or in general for $\mathbb{E}_1$ X, we have $BX \times X \simeq LBX$).

Pictorially, we have a circle for each integer, and S^1 acts by rotating the nth loop at n-times speed. So we might expect $\mathrm{THH}(\mathbb{S}[\mathbb{N}])$ to look like Σ^∞_+ applied something below:



and indeed, the actual answer is as follows:

$$\mathrm{THH}(\mathbb{S}[t]) \simeq \mathbb{S} \oplus \bigoplus_{n\geq 1} \Sigma^\infty_+ (S^1/C_n)$$

i.e. a wedge of a point and a bunch of circles, where the quotient $S^1/C_n \simeq S^1$ is meant to indicate the circle action (again, rotating the nth circle at n-times speed).

For a sketch of the actual calculation, we first consider

$$\mathrm{THH}(\mathbb{S}[\mathbb{N}]) \simeq \mathbb{S}[\mathbb{N}] \otimes_{\mathbb{S}[\mathbb{N}\times\mathbb{N}]} \mathbb{S}[\mathbb{N}] \simeq \mathbb{S}[\mathbb{N} \otimes_{\mathbb{N}\times\mathbb{N}} \mathbb{N}]$$

and then directly observe the geometric realization which defines the relative tensor product $\mathbb{N} \otimes_{\mathbb{N}\times\mathbb{N}} \mathbb{N}$:

$$|\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbb{N} \times (\mathbb{N}\times\mathbb{N}) \times \mathbb{N} \rightrightarrows \mathbb{N}\times\mathbb{N}|.$$

For ease of reading, let us denote this complex by

$$X = \cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} X_1 \rightrightarrows X_0.$$

Since the arrows preserve sums, we can decompose this geometric realization into a disjoint union indexed by $\mathbb{N}$, i.e. a disjoint union

$$\coprod_{n\in\mathbb{N}} |\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} X_1^{(n)} \rightrightarrows X_0^{(n)}|$$

where $X_m^{(n)}$ consists of the m-simplices which sum to n.

For example, for $n = 0$, there is only one way to sum to 0 in all degrees, and so we obtain

$$|\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} X_1^{(0)} \rightrightarrows X_0^{(0)}| = |\cdots \to * \to *| = *.$$

Meanwhile for $n = 1$, we have a picture that looks like

$$|\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \quad * * * * \rightrightarrows \; * * \,|$$

where in degree 0, we have simplices $(1,0)$ and $(0,1)$ in $\mathbb{N}\times\mathbb{N}$, and then the four simplices in $\mathbb{N}^4$, two of which are nondegenerate, and so on. One can find that this produces S^1, where we have only four nondegenerate simplices in total (2 points and 2 lines). The rest similarly produce circles (where n should also be the degree after which simplices are entirely degenerate).

One can verify the circle action via the map $\mathrm{THH}(\mathbb{S}[\mathbb{N}]) \to \mathrm{THH}(\mathbb{S}[\mathbb{Z}])$.

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