# EXAMPLES OF RING SPECTRA IN CHROMATIC HOMOTOPY THEORY

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Abstract. This document contains notes for European Talbot 2025, Lecture 6. Within it, we provide a basic introduction to several spectra important to chromatic homotopy theory, and we discuss constructions of their multiplicative structures. In particular, we construct Morava $E$-theory $E_n$ as an $\mathbb{E}_\infty$-ring, the Brown-Peterson spectrum $BP$ as an $\mathbb{E}_2$-ring, and the $\mathbb{E}_1$-$MU$-algebra structure on Morava $K$-theory $K(n)$.

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Our goal in this write-up is to talk about three major players in chromatic homotopy theory: Morava $E$-theory $E_n$, the Brown-Peterson spectrum $BP$, and Morava $K$-theory $K(n)$. Not only will this document introduce these spectra (with the exception of of $K(n)$, which was introduced and shown to be homotopy associative in Lecture 5), we will describe some of the methods used to construct multiplicative structures on spectra.

In particular, our discussion of $E_n$ will involve seeing how one might construct an $\mathbb{E}_\infty$ structure on a spectrum. We will mainly summarize results of Ramzi [Ram23], where we see that nice properties of $E_n$ allow us to lift results about it in the homotopy category (e.g. homotopy commutativity) to spectra. There is a separate currently unpublished proof by Robert Burkland, Dustin Clausen, and Ishan Levy, of which we copy down a brief sketch given by Ishan (for future reference).

Meanwhile, the other two spectra $BP$ and $K(n)$ are not $\mathbb{E}_\infty$, and so arguments about them will follow from a different type of approach. For these, to get $\mathbb{E}_1$ structures, we will discuss how multiplicative structures behave on quotients as in work of Angeltveit [Ang08], and we will also discuss how to get the $\mathbb{E}_2$-structure on $BP$ via a variation on an argument due to Chadwick-Mandell [CM16] provided to the author by Ryan Quinn.

## 1. Morava $E$-Theory

1.1. **Motivation.** The classification of formal group laws tells us that the moduli stack $\mathcal{M}_{fg}^{=n}$ of formal groups of height exactly $n$ has a unique geometric point, meaning that over an algebraically closed field, there's only one formal group up to isomorphism at height $n$. And Morava $K$-theory $K(n)$ corresponds to this geometric point.

But Morava $K$-theory was not so simple to define; we couldn't get it from Landweber exactness (indeed, the inclusion of a geometric point is unlikely to be flat). If instead we consider a formal neighborhood around this point (i.e. we consider small deformations of it), then we can look at the inclusion of this neighborhood into $\mathcal{M}_{fg}$, which should be flat, and thus Landweber exact.

This neighborhood is also conceptually important to study, because it is possible for formal groups not to be exactly of any height (when they are not over a field—e.g. by having nilpotent coefficients). We need to understand how the strata $\mathcal{M}_{fg}^{=n}$ are glued to each other in order to better understand $\mathcal{M}_{fg}$ as a whole, so we slightly deform $\mathcal{M}_{fg}^{=n}$ and see what happens.

In doing so, by Landweber exactness, we produce a new interesting cohomology theory: Morava $E$-theory, which we will discuss below.

1.2. **Initial construction.** To look at a neighborhood of $\mathcal{M}_{fg}^{=n}$, we consider rings that are "almost" fields, i.e. fields up to some nilpotent fuzz. Let $k$ be a perfect field of characteristic $p$.

**Definition 1.1.** An *infinitesimal thickening* of $k$ is a ring $A$ with a surjection $p : A \to k$ such that for $\mathfrak{m} = \ker(p)$, we have that

(1) $\mathfrak{m}^i = 0$ for $i$ large enough, and
(2) $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite dimensional $k$-vector space for every $i$.

**Example 1.1.** $A = k[\varepsilon]/(\varepsilon^2)$.

Suppose that $\mathbb{G}_0 \to \mathrm{Spec}(k)$ is a formal group over $k$ of height $n$. We want a definition that describes formal group laws that are "close" to $\mathbb{G}_0$.

**Definition 1.2.** A *deformation* of $\mathbb{G}_0$ to an infinitesimal thickening $A$ over $k$ is a formal group $\mathbb{G} \to \mathrm{Spec}(A)$ with an isomorphism

$$\mathbb{G} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \simeq \mathbb{G}_0,$$

i.e. $\mathbb{G}_0$ is the pullback of $\mathbb{G}$ along $p : A \to k$.

If we look at a formal group law $F_0 \in k[[x, y]]$ corresponding to $\mathbb{G}_0$, then $F$ would be a deformation of $F_0$ to $A$ if $F = F_0 \mod \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal given by $\ker(p : A \to k)$.

**Example 1.2.** Consider a formal group law $F$ over $A = k[\varepsilon]/(\varepsilon^2)$ defined by

$$[p]_F(x) = \varepsilon x^p + x^{p^2} + \dots$$

Let $F_0$ be a formal group law that is the reduction of $F \mod (\varepsilon)$. Then $F_0$ looks like

$$[p]_F(x) = x^{p^2} + \dots$$

and is exactly height 2. And $F$, which is not height exactly 2 (or height exactly $n$ for any $n$), is a deformation of $F_0$.

Let $\mathrm{Def}_{\mathbb{G}_0}(A)$ be the groupoid of deformations of $\mathbb{G}_0$ to $A$ with isomorphisms (which is actually discrete at finite height). Then we have the following theorem, which we will not prove:

**Theorem 1.1.** Let $k$ be a perfect field of characteristic $p$, and let $\mathbb{G}_0 \to \mathrm{Spec}(k)$ be a formal group of height $n < \infty$. Let $E_0 = W(k)[[u_1, \dots, u_{n-1}]]$ where $W(k)$ is the ring of Witt vectors of $k$. Then there exists a formal group $\mathbb{G} \to \mathrm{Spf}(E_0)$ which is a "universal deformation" in the sense that it induces the isomorphism

$$\mathrm{Spf}(E_0)(A) \simeq \mathrm{Def}_{\mathbb{G}_0}(A)$$

for any infinitesimal thickening $A$ of $k$.

In coordinates, let $F_0$ be a formal group law corresponding to $\mathbb{G}_0$. Then a universal deformation $F \in E_0[[x, y]]$ (corresponding to $\mathbb{G}$) is a formal group law which reduces to $F_0$ under the map $E_0 \to k$ and has $v_i(F) = u_i$, $1 \leq i \leq n-1$. In other words, it looks like

$$[p]_F(x) = px + u_1 x^p + \dots + u_{n-1} x^{p^{n-1}} + \text{something like } [p]_{F_0}(x).$$

The sequence $p, v_1, \dots, v_{n-1}$ is regular by construction in $E_0$, and $v_n(\mathbb{G})$ is a unit because $\mathbb{G}_0$ is height $n$. Thus, by the Landweber exact functor theorem, we have the following:

**Definition 1.3.** Fix $k$ and a formal group $\mathbb{G}$ of height $n$. By Landweber exactness, there is an even periodic, homotopy commutative homotopy ring spectrum denoted by $E = E(k, \mathbb{G}) = E_n$ whose homology theory is given by

$$E_*(X) = MU_*(X) \otimes_{MU_*} W(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$$

where $|u| = 2$. This is called the *nth Morava E-theory* or the Lubin-Tate spectrum. Sometimes implicitly $E$ is chosen with $\mathbb{G}$ as the Honda formal group and $k$ as $\mathbb{F}_p$.

We won't discuss the statement/proof of Landweber exactness here. But because this lecture is about multiplicative structures on spectra, we will briefly describe why Landweber exactness lands in homotopy commutative

spectra, particularly because it was not so immediate to see that $K(n)$ is in general homotopy associative/commutative (as it is, of course, not Landweber exact).

In summary, the homotopy commutative structure arises from the fact the product on homology is commutative, as the homology theory is defined by a tensor product of commutative rings. The obstruction to homotopy commutativity comes in the form of a phantom map

$$\mu - \mu \circ \tau : E \otimes E \to E.$$

Then there is a result of Hovey-Strickland [HS99, Cor. 2.15, Prop. 2.18] that phantom maps from evenly generated spectra (loosely, spectra that can be built from even finite spectra) to even spectra are automatically nullhomotopic, and one finds that tensor powers of Landweber exact spectra concentrated are evenly generated (which comes from the fact that $MU$ is evenly generated).

## 1.3. **Separability and the $\mathbb{E}_\infty$ structure.**

1.3.1. *Idea.* Now that we have the basic construction of Morava $E$-theory, let's discuss how one obtains an $\mathbb{E}_\infty$-ring structure on it.

Originally, this was done by Goerss-Hopkins-Miller via a lot of obstruction theory. The construction we'll be talking about today is that of Ramzi's [Ram23], which is a bit simpler to state and describe.

The basic idea is as follows: there is a nice property of certain algebras called separability. If an algebra $A \in \mathcal{C}$ is separable, then you can expect certain properties it has in the homotopy category $\mathrm{ho}(\mathcal{C})$ to lift to $\mathcal{C}$. To give a specific example, if $A$ is a separable $\mathbb{E}_1$-algebra and $\mathcal{C}$ satisfies some nice conditions, you might expect something like

$$\begin{array}{c}\text{homotopy commutative} \\ A \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{C}))\end{array} \quad \Longrightarrow \quad \begin{array}{c}\text{commutative} \\ A \in \mathrm{CAlg}(\mathcal{C})\end{array}$$

to be true, where the $\mathbb{E}_\infty$ structure extends the existing $\mathbb{E}_1$ structure.

But unfortunately, Morava $E$-theory turns out not to be separable; it is only *homotopy ind-separable*. Luckily, this ends up still working out, because of things like the fact that $\mathrm{Sp}_{K(n)}$ is compactly generated.

1.3.2. *Definitions.* Let's now discuss some background and terminology necessary to understand a summary of the argument. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal $\infty$-category.

**Definition 1.4.** An algebra $A \in \mathrm{Alg}(\mathcal{C})$ is called *separable* if the multiplication map $A \otimes A^{\mathrm{op}} \to A$ admits a section $s$ as a map of $A \otimes A^{\mathrm{op}}$-modules.

Similarly, $A$ is called *homotopy separable* if it is separable as an algebra in $\mathrm{ho}(\mathcal{C})$.

**Definition 1.5.** Now let $\mathcal{C}$ be compactly generated presentably, stably symmetric monoidal $\infty$-category. Let $A \in \mathrm{CAlg}(\mathcal{C})$. Then $A$ is *ind-separable* if there is a set $S \subseteq \pi_0(\mathbf{1}, A \otimes A)$ such that the multiplication map

$$A \otimes A \to A \simeq (A \otimes A)[S^{-1}]$$

witnesses $A$ a a localization in $A \otimes A$-modules.

A homotopy commutative homotopy algebra $A \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{C}))$ is called *homotopy ind-separable* if it is ind-separable in ho$\mathcal{C}$. Moreover, this turns out to be equivalent to there being a set $S \subseteq \pi_0 \operatorname{map}(\mathbf{1}, A \otimes A)$ such that the multiplication map $A \otimes A \to A$ an isomorphism

$$\pi_*(\operatorname{map}(c, A \otimes A))[S^{-1}] \to \pi_* \operatorname{map}(c, A)$$

for each compact object $c \in \mathcal{C}^\omega$.

Let's discuss an example, which will also provide some motivation for the approach to $E_n$.

Recall the following definition of Galois extension:

**Definition 1.6.** Let $G$ be a finite group, $R$ a commutative ring, and $B$ a commutative $R$-algebra with $G$-action. Then we say that $R \to A$ is a *$G$-Galois extension* if we have the equivalences

(1) $R \xrightarrow{\sim} A^G$ induced by the unit
(2) $A \otimes_R A \xrightarrow{\sim} \prod_G A$ induced by $a \otimes b \mapsto (a \cdot g(b))_g$.

Here is a fact:

**Fact 1.1.** Let $R \to A$ be a $G$-Galois extension, with $G$ a discrete group. Then $R \to A$ is separable.

In our case, there is a fact that we can view $E_n$ as a a $K(n)$-local (pro)-Galois extension of $L_{K(n)}\mathbb{S}$. This indicates to us that we should not expect separability, but rather some appropriate meaning of ind-separability.

**Theorem 1.2** ([Ram23] Cor. 4.37)**.** Let $\mathcal{C}$ satisfy the following assumptions:

(1) $\mathcal{C}$ is a compactly generated, stably symmetric monoidal $\infty$-category in which tensor products commute with colimits in each variable, and $\mathcal{C}^\omega$ is closed under non-empty tensor products.
(2) If $X, Y \in \mathcal{C}$ and there exists an isomorphism $f : \pi_0 \operatorname{Map}(-, X) \cong \pi_0 \operatorname{Map}(-, Y)$ between the cohomology theories they represent on $\mathcal{C}^\omega$, then there exists a map $\tilde{f} : X \to Y$ which lifts $f$ and is therefore an equivalence.

Let $A \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{C}))$ be a homotopy commutative, homotopy ind-separable homotopy algebra in $\mathcal{C}$ which receives no phantom maps from any tensor power of $A$. Then for any $1 \leq d \leq \infty$, the moduli space

$$\mathrm{Alg}_{\mathbb{E}_d}(\mathcal{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathcal{C}))^{\simeq}} \{A\}$$

is contractible; in other words, $A$ admits a unique lift to an $\mathbb{E}_d$-algebra in $\mathcal{C}$.

1.3.3. *The theme of the approach.* Let's go through and provide a very basic, intuitive (and overly simplistic) summary of the overarching ideas behind this argument. The first idea is that separability is like a projectivity condition, or more precisely, an algebra $A$ being separable is equivalent to its being projective as an $A \otimes A^{\mathrm{op}}$ module. Furthermore, if $A$ is separable, then every $A$-module is a retract of a free module $A \otimes M$ for some $M$.

This means that derived mapping spaces roughly end up being discrete or at least simply connected, so in some sense, they collapse to the hom-sets of the homotopy category. Conceptually, the analogy of this argument in obstruction theory calculations is that Ext-type terms end up vanishing due to projective or flat things.

This allows us to see the slogan that "the behavior of separable algebras is controlled by the homotopy category". For instance, one finds that $\mathrm{map}_{\mathrm{Alg}(\mathcal{C})}(A^{\otimes n}, A)$ is discrete and

$$\mathrm{map}_{\mathrm{Alg}(\mathcal{C})}(A^{\otimes n}, A) \simeq \hom_{\mathrm{Alg}(\mathrm{ho}(\mathcal{C}))}(hA^{\otimes n}, hA),$$

which means that $\mathbb{E}_d$ structures on $A$ extending the $\mathbb{E}_1$ strucure are equivalent to $\mathbb{E}_d$ structures on $hA$ in $\mathrm{Alg}(\mathrm{ho}(\mathcal{C}))$.

Similar arguments occur in the ind-case, with more complicated obstruction theory required, particularly the machinery of [PV22]; there is more stuff to handle, like phantom map problems (which arise from the fact that conditions are tested on compact objects).

1.3.4. *Why $\mathrm{Sp}_{K(n)}$ satisfies the assumptions.* Let us now describe why $\mathrm{Sp}_{K(n)}$ satisfies the assumptions of the main theorem, much of which we have seen before. For instance, certainly it is stably symmetric monoidal, and tensor products commute with colimits in each variable. Furthermore, $\mathrm{Sp}_{K(n)}$ is compactly generated by $L_{K(n)}X$ for any finite spectra $X$ of type $n$. And its compacts are dualizable, so they are closed under non-empty tensor products.

As for Assumption (2), it turns out this assumption is often satisfied in practice, and that it suffices for $\mathrm{ho}(\mathcal{C}^\omega)$ to have countably many isomorphism classes and countable mapping sets (being countable means $\mathrm{ho}(\mathcal{C}^\omega)$ turns out to be small enough that you can build up a spectrum from a cohomology theory by looking the cohomology of compact generators). This is then reduced to proving that $\pi_*(L_{K(n)}X)$ is countable for $X$ a finite spectrum (which is true by a spectral sequence argument).

1.3.5. *Why $E_n$ satisfies the hypotheses.* First, let's describe why $E_n$ is homotopy ind-separable. Ramzi's argument mainly relies on one external input: the computation of $E_n$'s ring of cooperations, i.e. $\pi_*(L_{K(n)}(E \otimes E))$. One sees that it can be identified with the graded ring of continuous functions on the Morava stabilizer group $C(\Gamma, E_*)$, and that the multiplication map

$$\pi_*(L_{K(n)}(E \otimes E)) \to E_*$$

is given by evaluation of functions on the identity $e \in \Gamma$. This ends up allowing us to see that the multiplication map witnesses $E$ as a localization $\pi_*(L_{K(n)}(E \otimes E))[S^{-1}]$ (where $S$ consists of indicator functions on nested open subgroups of $\Gamma$).

And lastly, the lack of phantom maps $E^{\otimes n} \to E$ comes from the same result about phantom maps between Landweber exact spectra as before.

One more thing we should mention is the fact that we want to see the $\mathbb{E}_\infty$ structure on $E_n$ inside of Sp, not $\mathrm{Sp}_{K(n)}$. This is okay because $E_n$ is already $K(n)$-local, and $\mathrm{Sp}_{K(n)}$ is a symmetric monoidal Bousfield localization of Sp.

So, for Morava $E$-theory $E = E(k, \mathbb{G})$, we have that for $1 \leq d \leq \infty$, the moduli space

$$\mathrm{Alg}_{\mathbb{E}_d}(\mathrm{Sp})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathrm{Sp}))^{\simeq}} \{E\}$$

is contractible, i.e. $E$ admits a unique $\mathbb{E}_d$ structure in spectra.

1.4. **An unpublished proof.** The last thing we do for Morava $E$-theory in this document is copy down an unpublished construction of $E$ as an $E_\infty$ ring. This sketch was sent to the author by Ishan Levy and is effectively copied down verbatim. The construction is upcoming work of Burklund, Clausen, and Levy.

Start by observing that for any formal group of height $n$ over a perfect ring $R$, the induced map $\mathrm{Spec}(R) \to \mathcal{M}_{fg}^{=n}$ is formally etale. This follows from the fact that the moduli of formal groups of height $n$ is essentially the classifying stack of a pro-finite etale group scheme (see Lurie's chromatic notes, Lecture 13, Theorem 11).

Then observe that formally etale algebras uniquely lift along square zero extensions of stacks (which is a version of Prop 3.35 of Lurie's DAG IV). $\mathcal{M}_{fg}^{=n}$ can be viewed as a locally closed substack of the category of sunthetic spectra, viewed as a spectral stack.

Indeed, it is the locus where $\tau, p, \ldots, v_{n-1} = 0$ and $v_n$ is invertible. Because the closed subset that it is an open inside of is cut out by a regular sequence, the fact about lifting along square zero extensions gives a unique lift of $R$ to a formal neighborhood of $M_{fg}^{=n}$ inside the larger stack.

This gives an algebra that is the connective cover of Morava $E$-theory: one way to see this is to note that it is complete with respect to $\tau, p, ..., v_{n-1}$ and agrees with $R$ after quotienting by this sequence, which shows via a Bockstein spectral sequence that it is even and has the desired homotopy ring and formal group.

## 2. The Brown-Peterson spectrum

2.1. **Construction.** One begins with looking at $MU_{(p)}$, the $p$-localization of $MU$. Recall that

$$\pi_*(MU_{(p)}) \simeq \mathbb{Z}_{(p)}[v_i \mid i \geq 1] \otimes \mathbb{Z}_{(p)}[b_m \mid m \neq p^k - 1]$$

where $|v_i| = 2p^i - 2$. In seeing the $v_i$'s pop up frequently and being useful to us, we may find it tempting to try to look only at them and discard the other stuff. Hence we obtain the $MU$-module $BP$ via

$$BP := MU_{(p)}/(b_m \mid m \neq p^k - 1)$$

where we have that

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots].$$

Then actually, we have the $MU_{(p)}$ decomposes as a spectrum into a direct sum of shifts of $BP$

$$MU_{(p)} \simeq \bigoplus_m \Sigma^{2m} BP$$

where $m$ is indexed over degrees of the monomials generated by $b_m$.

Another important construction is that of Quillen's, who constructed $BP$ as the image of an idempotent homotopy ring map

$$\epsilon : MU_{(p)} \to MU_{(p)}.$$

One can see that the ring homomorphism $MU_* \to MU_{(p)*} \to BP_*$ classifies the universal $p$-typical formal group law; as in, in the same way that $MU_*$ classifies formal group laws, $BP_*$ classifies $p$-typical ones. (Reminder: a $p$-typical formal group law is one whose logarithm has nonzero coefficients only for $p$-powers.)

In general, it is often fine to replace $MU$ with $BP$ in various places if you are at a prime $p$, and this can make things nicer because you only have to worry about the generators you care about.

2.2. **The $\mathbb{E}_1$ structure.** $BP$ is also Landweber exact, and so it is already homotopy associative/commutative (by the same arguments as before). We will now present a general result of Angeltveit that allows us to see how one can get an $\mathbb{E}_1$ structure in nice settings:

**Theorem 2.1** ([Ang08], Cor. 3.7)**.** Let $A = R/I$, where $R$ is an even $\mathbb{E}_\infty$-ring and $I$ is an ideal generated by a sequence of regular elements. Then if $A$ is homotopy associative, its homotopy associativity extends to an $\mathbb{E}_1$-$R$-algebra structure.

Angeltveit proves this with obstruction theory calculations; it's the kind of thing where you see the obstructions lie in $\mathrm{Ext}_{\pi_*(A \otimes_R A^{\mathrm{op}})}(A_*, A_*)$, and they end up vanishing for degree reasons, yadda yadda.

We apply this theorem to $R = MU$ and $I = (b_m \mid m \neq p^k - 1)$ and obtain that $BP$ has an $\mathbb{E}_1$ structure.

There is a lot of work relevant to this result, as well as extensions of it; see Hahn-Wilson [HW18], who strengthen this result (by showing that it suffices for $R$ to be $\mathbb{E}_2$) and give an account of relevant work in the introduction.

2.3. **The $\mathbb{E}_2$ structure.** Now we move onto seeing that there exists an $\mathbb{E}_2$ structure on $BP$. This proof actually does not use the $\mathbb{E}_1$ structure on $BP$ that we just talked about, and rather uses only its homotopy ring structure. (It actually turns out that this is not the highest degree of multiplicativity that is possible; see Lecture 10.5 or ask Ryan Quinn about it.)

In particular, we're interested in the following result of Chadwick-Mandell:

**Theorem 2.2** ([CM16] Cor. 1.3)**.** The Quillen idempotent $MU_{(p)} \to MU_{(p)}$ and the Quillen map $MU \to BP$ are represented by maps of $\mathbb{E}_2$-ring spectra.

Let's summarize the overarching ideas of this. Firstly, because $MU$ is a Thom spectrum, there exists an $\mathbb{E}_2$-map $\mathrm{id}_{MU} : MU \to MU$ (see Cor. 3.18, [ACB19], ) and the Thom isomorphism is multiplicative, we have that

$$\begin{aligned}\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})}(MU, MU) &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})}(\Sigma^\infty_+ BU, MU)\\ &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(BU, \Omega^\infty MU)\\ &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(BU, GL_1(MU))\\ &\simeq \mathrm{Map}_{\mathcal{S}}(B^2BU, B^2GL_1(MU))\\ &\simeq \mathrm{Map}_{\mathcal{S}}(BSU, B^2GL_1(MU))\end{aligned}$$

where the second equivalence is by loops-suspension, and the last one is by the fact that the group of units functor is right adjoint to the inclusion of grouplike $\mathbb{E}_n$-algebras in spaces into $\mathbb{E}_n$-spaces (and grouplike $\mathbb{E}_n$-algebras is a full subcategory of $\mathbb{E}_n$-algebras).

In general one also has the description that $\mathbb{E}_2$-maps $MU \to R$ are given by $\mathbb{E}_2$-nullhomotopies of the map $BU \to BGL_1(R)$.

Meanwhile, one can also use the fact that homotopy ring maps $MU \to MU$ are the same as complex orientations of $MU$, that is, maps of pointed spectra $MU(1) \to MU$, and apply the Thom isomorphism again:

$$\begin{aligned}\mathrm{Map}_{\mathrm{Alg}_{\mathrm{ho}(\mathrm{Sp})}}(MU, MU) &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Sp}_*}}(MU(1), MU)\\ &\simeq \mathrm{Map}_{\mathcal{S}_*}(\Sigma^{-2}\mathbb{C}P^\infty, GL_1(MU))\end{aligned}$$

and now then one does calculations to find that there exists no obstruction to lifting the Quillen idempotent (before $p$-localizing) to an $\mathbb{E}_2$ map (because of reasons like the fact that $MU$ is even, and the homology of $BSU$ is concentrated in even degrees). The concept is that one finds that the forgetful functor ends up being surjective on homotopy groups.

Then we have that $MU_{(p)} \to MU_{(p)}$ lifts to an $\mathbb{E}_2$ map, and so

$$BP = \mathrm{colim}(MU_{(p)} \xrightarrow{\epsilon} MU_{(p)} \xrightarrow{\epsilon} MU_{(p)} \longrightarrow \cdots)$$

has an $\mathbb{E}_2$ structure as well.

Chadwick-Mandell actually prove that in fact, if $R$ is an even $\mathbb{E}_2$-ring spectrum, every map of homotopy ring spectra $MU \to R$ lifts to a map of $\mathbb{E}_2$-ring spectra $MU \to R$. But the proof we described is simpler and specific

to $R = MU$, as it uses the fact that $\mathrm{Map}_{\mathbb{E}_2}(MU, MU)$ is nonempty. In the general case, they have to do a complicated fancy inductive argument.

## 3. Morava $K$-Theory

In the last lecture, we saw the construction of $K(n)$ as a homotopy ring spectrum. We can again use the result of Angeltveit to find that the homotopy associativity of $K(n)$ extends to an $\mathbb{E}_1$-ring structure.

In particular, applying Angeltveit to $R = MU$ and $I = (v_i, b_m \mid i \neq v_n, m \neq p^k - 1))$, we obtain that $k(n)$ has an $\mathbb{E}_1$-$MU$-algebra strructure.

Then the last thing to do would be to show that $K(n) \simeq v_n^{-1}k(n)$ is still $\mathbb{E}_1$. The concept here is that morally, inverting $v_n$ here does not mess up multiplicativity, in the same way localization does not mess up multiplicativity in classical algebra.

Unfortunately, the proof is rather involved and formal (according to Ishan, it has something to do with moving in to the category of $k(n)$ modules, but I have forgotten), so we will merely state the fact that: inverting elements in $\mathbb{E}_d$-rings is always okay ([MNN15], A.1).

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