TWO DETAILED COMPUTATIONS OF $\mathrm{THH}(\mathbb{F}_p)$

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Abstract. In the following expository document, we present two different proofs of Bokstedt periodicity: first, we explain an algebraic argument using group rings, and then we present a more topological argument via Thom spectra. Lastly, we describe a comparison of the two.

Contents

Our goal in this write-up is to explain two proofs of the following theorem:

Theorem 0.1 (Bökstedt)**.** We have that

$$\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[x],\ |x| = 2.$$

In particular, we have the following equivalence of $\mathbb{E}_1$-algebras:

$$\mathrm{THH}(\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3.$$

The proofs will be of different flavors: one more algebraic and one more topological. The former originates in a paper of Krause and Nikolaus [KN19], and the latter is given in greater generality by [BCS10] (a more detailed discussion of this particular computation found in [KN18]).

The aim of this write-up is to go through every step as carefully as possible, even if it seems obvious or trivial to the reader (because it definitely wasn't obvious or trivial to me). I may have still missed some parts, but I hope to be more explicit than everything I have read so far.

In this document, we will use the following definition of THH:

Definition 0.1. Let R be an associative ring spectrum. Then we define

$$\mathrm{THH}(R) = R \otimes_{R \otimes R^{\mathrm{op}}} R.$$

When R is a classical ring, we will use the notation $\mathrm{THH}(R) := \mathrm{THH}(HR)$, where HR is the Eilenberg-MacLane spectrum of R.

Tensor (smash) products of spectra will be over the sphere spectrum $\mathbb{S}$ unless otherwise noted. Also, we will freely use the language and ideas of higher algebra; we will denote the ∞-category of spaces by $\mathcal{S}$ and the symmetric monoidal stable ∞-category of spectra by Sp. Other definitions will be introduced as they become necessary.

1. The dual Steenrod algebra as a free object

In some sense, computing $\mathrm{THH}(\mathbb{F}_p)$ is equivalent to understanding the dual Steenrod algebra $\pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p) = H_*(H\mathbb{F}_p; \mathbb{F}_p)$, i.e. the lower tensor product in our definition of THH. The two proofs we present in these notes both reduce to proving the following result:

Theorem 1.1. There is an equivalence of $\mathbb{E}_2$-ring spectra

$$H\mathbb{F}_p \otimes H\mathbb{F}_p \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3.$$

Stated in words, this just means: "As an $\mathbb{E}_2$-ring, the dual Steenrod algebra is the free $H\mathbb{F}_p$-algebra on a single generator in degree 1." We'll now explain why.

Let us denote

$$\mathbb{F}_p[-] := H\mathbb{F}_p \otimes \Sigma^\infty_+(-),$$

which is a functor from $\mathcal{S}$ to $\mathrm{Mod}_{H\mathbb{F}_p}$. This is an example of a more general construction:

Definition 1.1. Let G be an ∞-group, and R a commutative ring spectrum. The *group ring* $R[G]$ is the R-algebra spectrum $R \otimes \Sigma^\infty_+ G$.

In our case, we can examine exactly in what sense $\mathbb{F}_p[\Omega^2 S^3]$ is the free $\mathbb{E}_2$-$H\mathbb{F}_p$-algebra on a single degree 1 generator: let A be another $\mathbb{E}_2$-$H\mathbb{F}_p$-algebra, and let

$$\mathrm{Alg}^{\mathbb{E}_2}_{H\mathbb{F}_p} := \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{H\mathbb{F}_p})$$

be the category of $\mathbb{E}_2$-$H\mathbb{F}_p$-algebras. Since $H\mathbb{F}_p$ is commutative, this is equivalent to the category of $\mathbb{E}_2$-ring spectra A with an $H\mathbb{F}_p$ action $H\mathbb{F}_p \to A$ [Lur17, 7.1.3.8.]. Then we have that:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{H\mathbb{F}_p}}(\mathbb{F}_p[\Omega^2 S^3], A) &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Sp})}(\Sigma^\infty_+ \Omega^2 S^3, A) \\
&\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(\Omega^2 S^3, \Omega^\infty A) \\
&\simeq \mathrm{Map}_{\mathcal{S}_*}(S^1, \Omega^\infty A)
\end{aligned}$$

through a chain of adjunctions. In particular, we pass through the free-forgetful adjunction $\mathrm{Sp} \rightleftharpoons \mathrm{Mod}_{H\mathbb{F}_p}$, suspension-loops $\Sigma^\infty_+ \dashv \Omega^\infty$, and the free $\mathbb{E}_n$-space construction $X \to \Omega^n\Sigma^n X$. In other words, elements of

$$\pi_0 \operatorname{Map}_{\mathcal{S}_*}(S^1, \Omega^\infty A) \cong \pi_1(A)$$

correspond exactly to homotopy classes of $H\mathbb{F}_p$-linear $\mathbb{E}_2$ maps

$$\mathbb{F}_p[\Omega^2 S^3] \to A,$$

in a coherent way; i.e. $\mathbb{F}_p[\Omega^2 S^3]$ is universally determined by a single generator in degree 1 as an $\mathbb{E}_2$-$H\mathbb{F}_p$-algebra. In our case, we let $A = H\mathbb{F}_p \otimes H\mathbb{F}_p$ (where the $H\mathbb{F}_p$ action is by inclusion into the left factor).

Assume that we do prove Theorem 1.1, i.e. that $H\mathbb{F}_p \otimes H\mathbb{F}_p \simeq \mathbb{F}_p[\Omega^2 S^3]$. Then the final computation of THH follows almost immediately, as seen below:

1.1. **After this, we're done!** Note that $\mathbb{F}_p[-]$ preserves colimits because $H\mathbb{F}_p \otimes -$ and $\Sigma^\infty_+(-)$ are both left adjoints. It also sends products to tensor products, since both $H\mathbb{F}_p \otimes -$ and $\Sigma^\infty_+(-)$ are symmetric monoidal. Then we have that

$$\begin{aligned}
\mathrm{THH}(\mathbb{F}_p) &= H\mathbb{F}_p \otimes_{H\mathbb{F}_p \otimes H\mathbb{F}_p} H\mathbb{F}_p \\
&\simeq H\mathbb{F}_p \otimes_{\mathbb{F}_p[\Omega^2 S^3]} H\mathbb{F}_p \\
&\simeq (H\mathbb{F}_p \otimes \mathbb{S}) \otimes_{\mathbb{F}_p[\Omega^2 S^3]} (H\mathbb{F}_p \otimes \mathbb{S}) \\
&\simeq \mathbb{F}_p[*] \otimes_{\mathbb{F}_p[\Omega^2 S^3]} \mathbb{F}_p[*] \\
&\simeq \operatorname{colim}(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} (\mathbb{F}_p[*] \otimes \mathbb{F}_p[\Omega^2 S^3] \otimes \mathbb{F}_p[*]) \rightrightarrows (\mathbb{F}_p[*] \otimes \mathbb{F}_p[*])) \\
&\simeq \operatorname{colim}(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathbb{F}_p[* \otimes \Omega^2 S^3 \otimes *] \rightrightarrows \mathbb{F}_p[* \otimes *]) \\
&\simeq \mathbb{F}_p\left[\operatorname{colim}(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} (* \otimes \Omega^2 S^3 \otimes *) \rightrightarrows *)\right] \\
&\simeq \mathbb{F}_p[B\Omega^2 S^3] \\
&\simeq \mathbb{F}_p[\Omega S^3] \\
&= H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3
\end{aligned}$$

as desired. In the end, we obtain an $\mathbb{E}_1$-equivalence; all maps preserved multiplicativity except the bar-loops equivalence, which decreased us to an $\mathbb{E}_1$ equivalence of ring spectra. Here we also used the fact that a model for the classifying space BG is given by the "colimit of a simplicial diagram" bar construction $\mathrm{Bar}(*, G, *)$.

And so THH($\mathbb{F}_p$) is a free $\mathbb{E}_1$-$H\mathbb{F}_p$-algebra on a generator in degree 2. Thus the question remains: how do we see that the dual Steenrod algebra has the free structure we want?

1.2. **Results of Araki-Kudo, Dyer-Lashof, and Steinberger.** One might guess that in order to prove that

$$H\mathbb{F}_p \otimes H\mathbb{F}_p \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3,$$

we ought to look explicitly at the $\mathbb{E}_2$ Dyer-Lashof operations on homology, and use them to show that

$$H_*(H\mathbb{F}_p;\mathbb{F}_p) \cong H_*(\Omega^2 S^3;\mathbb{F}_p),$$

especially since these things seem like they would be computable with very classical methods. Indeed, let us recall that for a $\mathbb{E}_2$-ring spectrum X, we have the Dyer-Lashof operations on $\mathbb{F}_p$ homology:

$$\begin{aligned} p = 2: \quad & Q^i : H_k(X;\mathbb{F}_2) \to H_{k+i}(X;\mathbb{F}_2) \\ p > 2: \quad & Q^i : H_k(X;\mathbb{F}_p) \to H_{k+2i(p-1)}(X;\mathbb{F}_p) \\ & \beta Q^i : H_k(X;\mathbb{F}_p) \to H_{k+2i(p-1)-1}(X;\mathbb{F}_p) \end{aligned}$$

which satisfy certain conditions, formulae, and relations such that they make sense (e.g. the Cartan formula and the Adem and Nishida relations.)

Then we have the following calculational results:

Lemma 1.1. (Araki-Kudo, Dyer-Lashof, and Steinberger.)

(1) For $p = 2$, we have that

$$H_*(\Omega^2 S^3;\mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, \ldots]$$

where $|x_i| = 2^i - 1$, and $x_{i+1} = Q^{2^i} Q^{2^{i-1}} \cdots Q^4 Q^2 x_1$, and $\beta x_i = x_{i-1}^2$; i.e., it is generated by x_1 over the Dyer-Lashof algebra.

(2) For $p > 2$, we have that

$$H_*(\Omega^2 S^3;\mathbb{F}_p) \cong \mathbb{F}_p[y_0, y_1, \ldots; z_1, z_2, \ldots]/(y_i^2)$$

where $|y_i| = 2p^i - 1$, and $y_{i+1} = Q^{p^i} \cdots Q^p Q x_1$; and $|z_i| = 2p^i - 2$ $z_i = \beta y_{i+1}$; i.e., it is generated by y_0 over the Dyer-Lashof algebra.

(3) The same isomorphisms hold for $\mathcal{A}_* = H_*(H\mathbb{F}_p;\mathbb{F}_p)$, though we use different symbols;

$$\begin{aligned} p = 2: \quad & \mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \\ p > 2: \quad & \mathcal{A}_* \cong \mathbb{F}_p[\tau_0, \tau_1, \ldots; \xi_1, \xi_2, \ldots]/(\tau_i^2) \end{aligned}$$

with the analogous relations; i.e. when $p = 2$, $\mathcal{A}_*$ is generated by ξ_1 over the Dyer-Lashof algebra, and when $p > 2$, $\mathcal{A}_*$ is generated by τ_0 over the Dyer-Lashof algebra.

The first result is [KA56, Theorem 7.1], which computes $H_*(\Omega^N S^n;\mathbb{F}_2)$ for all $N < n$. The second is [DL62, Theorem 5.2], which computes $H_*(\Omega^N S^n;\mathbb{F}_p)$ for all $N < n$ with n odd. The third is [BMMS86, Theorem 2.2, 2.3, 2.4].

The first two are applications of Serre's spectral sequence and transgressions, and the third is a combinatorial and inductive argument based on the Nishida relations and Milnor's original formulation (in terms of dual elements to Steenrod operations).

Let's give a *very* elementary explanation of how these were proven (especially since I had never really done spectral sequence computations before starting this project). In particular, let us just consider the case where $p = 2$, i.e. Araki-Kudo's calculation and Steinberger's Theorem 2.2 and 2.4.

Proof sketch of (1). To understand how one calculates the homology of our two-fold loop space, we will need to go through some basic facts on spectral sequences. One can first directly compute $H_*(\Omega S^3; \mathbb{F}_p) \cong \mathbb{F}_p[x]$, $|x| = 2$ by considering the standard path space fibration

$$\Omega S^3 \to P(S^3) \to S^3$$

where $P(S^3)$ is the based path space of S^3; in Serre's spectral sequence, this gives us

$$E^2_{p,q} = H_p(S^3; H_q(\Omega S^3)) \Rightarrow H_{p+q}(PS^3)$$

where PS^3 is contractible, so $H_{p+q}(PS^3)$ is trivial except in degree 0. Additionally, $E^2_{p,q} = H_q(\Omega S^3)$ if $p = 0, 3$ and it is 0 otherwise. Recall that the differential d^r is in bidegree $(-r, r-1)$. Then below we have pictures of the E^2 and E^3 pages, where we have abbreviated $H_q(\Omega S^3)$ by H_q.

The E^2 page

$$\begin{tikzcd}
q & & & & \\
3 & H_3 & 0 & 0 & H_3 \\
2 & H_2 & 0 & 0 & H_2 \arrow[ull] \\
1 & H_1 & 0 & 0 & H_1 \arrow[ull] \\
0 & H_0 & 0 & 0 & H_0 \arrow[ull] \\
 & 0 & 1 & 2 & 3 & p
\end{tikzcd}$$

The E^3 page

$$\begin{tikzcd}
q & & & & \\
3 & H_3 & 0 & 0 & H_3 \\
2 & H_2 & 0 & 0 & H_2 \\
1 & H_1 & 0 & 0 & H_1 \arrow[uulll] \\
0 & H_0 & 0 & 0 & H_0 \arrow[uulll] \\
 & 0 & 1 & 2 & 3 & p
\end{tikzcd}$$

Since only d^3 can be nonzero (because the others never line up properly), E^4 is the limiting page. And since the spectral sequence converges to $H_*(PS^3)$ which is nontrivial only in degree 0, we have that d^3 must in fact consist of isomorphisms, so that it kills all the terms in E^4 except $E^4_{0,0}$. In other words, we have that $H_n(\Omega S^3) \cong H_{n+2}(\Omega S^3)$ for all n, and $H_0(\Omega S^3) \cong \mathbb{F}_p$, and $H_1(\Omega S^3) \cong 0$. Hence, $H_*(\Omega S^3) \cong \mathbb{F}_p[x]$, $|x| = 2$.

Now that we have done this very elementary spectral sequence computation (actually, this was the base case in Araki-Kudo, who prove the result by induction), let us attempt to compute $H_*(\Omega^2 S^3; \mathbb{F}_2)$ again via the path space fibration:

$$\Omega^2 S^3 \to P(\Omega S^3) \to \Omega S^3.$$

Now the spectral sequence is no longer so easy, since by our previous calculation, we have

$$E^2_{p,q} = H_p(\Omega S^3; H_q(\Omega^2 S^3)) \Rightarrow H_{p+q}(P\Omega S^3),$$

so $E^2_{p,q} \cong H_q(\Omega^2 S^3)$ for $p \geq 0$ even, 0 otherwise. So our differentials don't have an obvious pattern, since they might end up being nontrivial on a bunch of even pages. But this turns out to be exactly the right setting to apply Dyer-Lashof operations.

First, recall that the long exact sequence of homotopy groups for a fibration $F \to E \to B$ has connecting maps $\pi_n(B) \to \pi_{n-1}(F)$; to study homology groups of fibers, we can analogously attempt to define a map $H_n(B) \to H_{n-1}(F)$ via long exact sequences in homology of the pairs (E, F) and $(B, *)$. In particular, we get a map τ, called the *transgression*, which maps a subgroup of $H_*(B)$ to a quotient of $H_*(F)$. In the case of the Serre spectral sequence, we have that this map is exactly

$$d^n : E^n_{n,0} \to E^n_{0,n-1}.$$

Intuitively, since we're only in the first quadrant, these are the longest differentials that still end up on the leftmost column of the spectral sequence. This leftmost column usually consists of the desired homology groups of the fiber. The transgression is important because there's "nowhere to go afterwards", and so it often determines the structure of $H_*(F)$. For example, in the previous calculation, we knew that the transgression

$$d^3 : H_0(\Omega S^3) = E^3_{3,0} \to E^3_{0,2} = H_2(\Omega S^3)$$

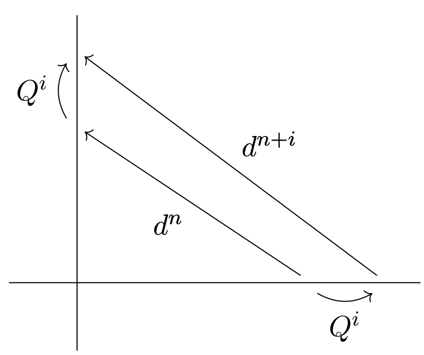
needed to be an isomorphism, and since we knew $H_0(\Omega S^3)$, this determined the entire result.

Let us call elements of $E^n_{n,0}$ *transgressive*; we similarly also call elements in $x \in E^2_{p,q}$ transgressive if $d^i(x) = 0$ for $2 \leq i < p$ and if its *transgression image* is nonzero, i.e. $\tau(x) = d^p(x) \neq 0$ in $E^p_{0,p+q-1}$.

Then mod 2 Dyer-Lashof operations commute with homomorphisms in long exact sequences of homology used in the construction of τ (which can be checked explicitly using properties of power operations). Moreover, the Q^i induce well-defined operations $E^n_{n,0} \to E^{n+i}_{n+i,0}$ and $E^n_{0,n-1} \to E^n_{0,n+i-1}$. Hence, the Dyer-Lashof operations commute with τ, i.e. we have

$$Q^i \circ \tau = \tau \circ Q^i,$$

as is represented pictorially in the following diagram:



So, if x is transgressive, so is $Q^J x$ for all admissible sequences J of Dyer-Lashof operations. In our case, we have that

$$d^2 : H_2(\Omega S^3) \cong E^2_{2,0} \to E^2_{0,1} \cong H_1(\Omega^2 S^3)$$

is an isomorphism; it sends the generator $x \in H_2(\Omega S^3)$ to its transgression image, a generator x_1 of $H_1(\Omega^2 S^3)$, and by Hurewicz, both sides are isomorphic to $\pi_3(S^3) \otimes \mathbb{F}_2 \cong \mathbb{F}_2$.

But then $Q^J x$ are also all transgressive, and as powers of x, they give a complete description of x as the generator of $H_*(\Omega S^3)$. Furthermore, transgression images should cover all generators in the leftmost column (i.e. $H_*(\Omega^2 S^3)$) by the time the sequence degenerates. Hence the transgression images

$$\tau(Q^J x) = Q^J(\tau(x)) = Q^J x_1$$

are the generators of $H_*(\Omega^2 S^3)$. In other words, $H_*(\Omega^2 S^3; \mathbb{F}_2)$ is generated by x_1 under the action of the mod 2 Dyer-Lashof operations, as desired.

This was not the most rigorous or complete argument, but hopefully it gives a picture of how transgressions are used to compute spectral sequences. For $p > 2$, the style of the argument is fairly similar, but the technical details are more complicated, so we will not discuss it. □

Proof summary of (3) for $p = 2$. After that long interlude on spectral sequences, let's give a brief summary of the main ideas used in Steinberger's calculation.

Recall that in Milnor's original paper, the mod 2 dual Steenrod algebra has the structure of the following graded polynomial algebra:

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

where $|\xi_i| = 2^i - 1$. (Milnor denotes ξ_i by ζ_i in the appendix where this result is stated.) Let $\bar{\xi}_i$ denote the conjugate or antipode of ξ_i in the Hopf algebra structure of $\mathcal{A}_*$.

Steinberger obtains the following two results:

(1) (Theorem 2.2.) For $i > 1$, we have that $Q^{2^i-2}\xi_1 = \bar{\xi}_i$, and

$$Q^s \bar{\xi}_i = \begin{cases} Q^{s+2^i-2}\xi_1 & \text{if } s \equiv 0, -1 \mod 2^i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $Q^{2^i}\bar{\xi}_i = Q_1(\bar{\xi}_i) = \bar{\xi}_{i+1}$.

(2) (Theorem 2.4.) We have the following identity on total classes:

$$(1 + \xi_1 + \xi_2 + \cdots)^{-1} = 1 + \xi_1 + Q^1\xi_1 + Q^2\xi_1 + \cdots.$$

Hence, these two results allow us to rewrite the above polynomial algebra as

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, Q_1\xi_1, (Q_1)^2\xi_1, \dots]$$

as desired (where we have used lower indexing notation; i.e. $Q_r x = Q^{r+|x|}x$).

To prove the first theorem, Steinberger mainly applies the following strategies: firstly, he proceeds by induction on s and/or i. Then he considers existing formulae which describe the interaction between the generators ξ_i and the dual Steenrod operations $P_r := Sq^r_*$, since it is sufficient to show that both sides of the identities agree after applying P_{2^k} for $k \geq 0$. For instance, we have that for $i > 0$,

$$P_r \bar{\xi}_1 = \begin{cases} \bar{\xi}_{i-k}^{2^k} & \text{if } r = 2^k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

And lastly, he applies the Nishida relations which relate these P_r to Dyer-Lashof operations: for $p = 2$, we have that

$$P_r Q^s = \sum_i \binom{2^n + s - r}{r - 2i} Q^{s-r+i} P_i$$

where n is chosen to be sufficiently large such that the combinations makes sense.

For example, to show that $Q^{2^i-2}\xi_1 = \bar{\xi}_i$, he argues by induction on i. By the Nishida relations and combinatorial arguments,

$$P_{2^k} Q^{2^i-2}\xi_1 = \binom{2^i - 2 - 2^k}{2^k} Q^{2^i-2-2^k}\xi_1$$

which is trivial except when $k = 0$. By the relations above and previous lemma stating that $Q^{2s-1}\xi_1 = (Q^{s-1}\xi_1)^2$, which he also proves by induction on s, he has that

$$P_1 Q^{2^i-2}\xi_1 = Q^{2^i-3}\xi_1 = (Q^{2^{i-1}-2}\xi_1)^2 = (\bar{\xi}_{i-1})^2 = P_1\bar{\xi}_i.$$

The remainder of the proof of Theorem 2.2 uses similar techniques.

Then for Theorem 2.4, he defines the algebra $C = \mathbb{F}_2[x, x^{-1}]$ as an algebra over the Steenrod algebra $\mathcal{A}$ which inverts the generator of $H^*(\mathbb{R}P^\infty)$, and lets C_* be its dual, with generators e_n in degree n. Then he considers the map $f : C_* \to \mathcal{A}_*$ given by

$$fe_n = Q^n\xi_1$$

for $n > 0$, ξ_1 for $n = 0$, 1 for $n = -1$, and 0 otherwise, and shows that it is a map of $\mathcal{A}_*$-coalgebras. In particular, he argues that f is the unique nontrivial map of $\mathcal{A}_*$ comodules such that $f(e_{-1}) = 1$, which implies that $f(e_n)$ is the degree $(n+1)$ component $(\xi^{-1})_{n+1}$ of the inverted total class ξ^{-1}, giving us the result. To do so, he uses the algebra homomorphism $\varphi : C \to C\hat{\otimes}\mathcal{A}_*$ (where the tensor product is completed due to infinite sums) induced by dualizing $\mathcal{A} \otimes C \to C$ twice, and applies Milnor's calculation that

$$\varphi(x) = \sum_{i\geq 0} x^{2^i} \otimes \xi_i = \sum_{i \geq 1} x^i \otimes (\xi)_{i-1}$$

and linearity to obtain $\varphi(x^{-1})$ in terms of the inverse the total class, i.e.

$$\varphi(x^{-1}) = \sum_{i\geq -1} x^i \otimes (\xi^{-1})_{i+1}.$$

But simultaneously, by the construction of f, one has that $f(e_i) = (\xi^{-1})_{i+1}$, because for all $a \in \mathcal{A}$

$$\langle a, f(e_i)\rangle = \langle ax^{-1}, e_i\rangle = \langle \langle a, (\xi^{-1})_{i+1}\rangle x^i, e_i\rangle = \langle a, (\xi^{-1})_{i+1}\rangle$$

where the second equality comes from the fact that if $\varphi(c) = \sum c_i \otimes \alpha_i$ for some $c \in C$, then $ac = \sum \langle a, \alpha_i\rangle c_i$. □

2. The algebraic argument

So, now that we've spent a long time going through the classical results that we needed, let's return to the original problem, i.e. showing that we have the $\mathbb{E}_2$-equivalence of ring spectra

$$H\mathbb{F}_p \otimes H\mathbb{F}_p \simeq \mathbb{F}_p[\Omega^2 S^3].$$

The bulk of the work for our algebraic argument was covered in the last section, so this will actually be very short!

Proof of Theorem 1.1. We know from the above calculation that

$$\pi_1(H\mathbb{F}_p \otimes H\mathbb{F}_p) \cong H_1(H\mathbb{F}_p; \mathbb{F}_p) \cong \mathbb{F}_p.$$

This can also be calculated explicitly with universal coefficients. Then from Section 2, by the universal property of $\mathbb{F}_p[\Omega^2 S^3]$, we know that the generator ξ_1 of $H_*(H\mathbb{F}_p; \mathbb{F}_p)$ in degree 1 induces a $\mathbb{E}_2$ map

$$f : \mathbb{F}_p[\Omega^2 S^3] \to H\mathbb{F}_p \otimes H\mathbb{F}_p$$

which is determined by the fact that it takes x_1 to ξ_1 in degree 1. Then this map induces the map

$$f_* : H_*(\Omega^2 S^3; \mathbb{F}_p) \to H_*(H\mathbb{F}_p; \mathbb{F}_p)$$

Yet the preceeding classical calculations stated that both rings are polynomial rings generated by applying $\mathbb{E}_2$-Dyer-Lashof operations to a single generator in degree 1. And since f was constructed as an $\mathbb{E}_2$ map and hence preserves $\mathbb{E}_2$-Dyer-Lashof operations on $\mathbb{F}_p$-homology, it is sufficient to see that $f_*(x_1) = \xi_1$, which is exactly what we have already done. □

3. The Hopkins-Mahowald result

The following section explains the more topological version of the proof of Theorem 1.1, which rests on a result shown by Mahowald for $p = 2$ and by Hopkins for $p > 2$. A general construction computing THH of Thom spectra is given by [BCS10]. Before we state this result, let's summarize our strategy for this section.

The general steps are as follows.

(1) Find a spectrum with "more understandable" $\mathbb{F}_p$ homology (in particular with the same $\mathbb{F}_p$-homology as $\Sigma^\infty_+ \Omega^2 S^3$) by constructing a map f_p on $\Omega^2 S^3$ and taking its generalized Thom spectrum Mf_p.

(2) Show that we have an $\mathbb{E}_2$-equivalence of ring spectra $Mf_p \simeq H\mathbb{F}_p$, which reduces to checking the following equivalence on $\mathbb{F}_p$-homology:

$$H\mathbb{F}_p \otimes Mf_p \simeq H\mathbb{F}_p \otimes H\mathbb{F}_p.$$

(3) Use our classical calculations of $\mathbb{F}_p$-homology and $\mathbb{E}_2$-Dyer-Lashof operations to make further reductions and hence show that the above equivalence holds.

3.1. **Generalized Thom spectra.** We will follow [ABG+14]. Our goal here is to define a construction which will be analogous to the Thom space associated to a line bundle; in particular, given a ring spectrum R and a space X, we will define an R-line bundle over X, and then define its Thom spectrum.

Let R-line $\subseteq \mathrm{Mod}_R$ be the full subcategory of free rank one R-modules, i.e. modules $M \in \mathrm{Mod}_R$ such that $M \simeq R$ is an equivalence in Mod_R. Then by [ABG+14, Corollary 1.14] this simplicial set is a model for the space $BGL_1(R)$, where $GL_1(R)$ is defined to be the following pullback:

$$\begin{tikzcd}
GL_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^\infty R \arrow[d] \\
(\pi_0\Omega^\infty R)^\times \arrow[r] & \pi_0\Omega^\infty R
\end{tikzcd}.$$

In other words, $GL_1(R)$ is a subspace of $\Omega^\infty R$, and an element in $GL_1(R)$ is given by an element of $\Omega^\infty R$ whose path component is a unit in $\pi_0(\Omega^\infty R)$. By definition, we have that $\pi_0(GL_1(R)) = \pi_0(R)^\times$. Intuitively, this is analogous to the fact that $GL_1(R) \cong R^\times$ for classical rings, and that there is a unique free rank one R-module up to isomorphism, with automorphisms given by units (so R-line should be like the nerve of $R^\times$).

Definition 3.1. Let R be a ring spectrum and X a space. Then a *bundle of R-lines over X* is a functor

$$f : X \simeq X^{\mathrm{op}} \to BGL_1(R).$$

Here we have passed under the ∞-categorical equivalence between spaces and simplicial sets induced by the singular complex functor, and by abuse of notation, we don't distinguish between a map of spaces f and a functor of infinity groupoids $\mathrm{Sing}(f)$. We will continue to do so in this document, and hopefully it doesn't mess anything up.

Definition 3.2. Let $f : X \to BGL_1(R)$ be a bundle of R-lines over X. Then the *generalized Thom spectrum Mf* of f is given by the colimit

$$Mf = \mathrm{colim}(X \xrightarrow{f} BGL_1(R) \simeq R\text{-line} \hookrightarrow \mathrm{Mod}_R).$$

It is a fact that if X is an $\mathbb{E}_n$ space and f is an $\mathbb{E}_n$ map, then Mf is an $\mathbb{E}_n$ ring spectrum [KN18, Lemma 4.5].

Let's give a couple of examples.

Example 3.1 (The trivial bundle)**.** Let $\underline{R} : X \to BGL_1(R)$ be the trivial bundle, i.e. it takes all morphisms to 1. In other words, the composite functor will simply be the constant functor $R : X \to \mathrm{Mod}_R$, and so

$$M\underline{R} = \operatorname*{colim}_X R \simeq \operatorname*{colim}_X(\Sigma^\infty_+(*) \otimes R) \simeq \Sigma^\infty_+(\operatorname*{colim}_X(*)) \otimes R \simeq \Sigma^\infty_+ X \otimes R.$$

Example 3.2 (Homotopy orbits)**.** If $X = BG$ for some $\mathbb{E}_1$ group G, then composing

$$BG \xrightarrow{f} BGL_1(R) \to \mathrm{Mod}_R$$

with the forgetful functor $\mathrm{Mod}_R \to \mathrm{Sp}$ gives a spectrum with G action $R : BG \to \mathrm{Sp}$. Then $Mf = R_{hG}$ by definition, since the forgetful functor $\mathrm{Mod}_R \to \mathrm{Sp}$ commutes with limits and colimits [Lur17, 4.2.3.3].

Remark 3.1. There is one more property that we will need, which is that one can "change fiber". In other words, given a map $r : R \to R'$, we have that as a colimit, $M(-)$ commutes with the extension of scalars $- \otimes_R R' : \mathrm{Mod}_R \to \mathrm{Mod}_{R'}$, and so

$$M(BGL_1(r) \circ f) \simeq Mf \otimes_R R'$$

is the colimit of the bottom row of the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & BGL_1(R) \arrow[r] \arrow[d, "BGL_1(r)"] & \mathrm{Mod}_R \arrow[d, "- \otimes_R R'"] \\
X \arrow[r] & BGL_1(R') \arrow[r] & \mathrm{Mod}_{R'}
\end{tikzcd}$$

3.2. **The statement.** Now let's actually give the Hopkins-Mahowald result. First, let us consider the following calculation:

$$\pi_1(BGL_1(\mathbb{S}_p^\wedge)) \cong \pi_0(\Omega BGL_1(\mathbb{S}_p^\wedge)) \cong \pi_0(GL_1(\mathbb{S}_p^\wedge)) \cong \pi_0(\mathbb{S}_p^\wedge)^\times \cong \mathbb{Z}_p^\times$$

where we used the fact that $\pi_0(\mathbb{S}_p^\wedge) \cong \mathbb{Z}_p$. Consider the element

$$1 - p \in \mathbb{Z}_p^\times \cong \pi_1(BGL_1(\mathbb{S}_p^\wedge)).$$

Let it be represented by a map $S^1 \to BGL_1(\mathbb{S}_p^\wedge)$ which induces a two-fold loop map

$$f_p : \Omega^2 S^3 \to BGL_1(\mathbb{S}_p^\wedge).$$

This map is an $\mathbb{E}_2$ map, since $\Omega^2\Sigma^2 S^1 \simeq \Omega^2 S^3$ is the free $\mathbb{E}_2$ algebra on S^1, e.g. we have

$$\mathrm{Map}_{\mathbb{E}_2}(\Omega^2\Sigma^2 S^1, BGL_1(\mathbb{S}_p^\wedge)) \simeq \mathrm{Map}_{\mathcal{S}_*}(S^1, BGL_1(\mathbb{S}_p^\wedge)),$$

or more generally, $\Omega^n\Sigma^n X$ is the free $\mathbb{E}_n$ algebra on a connected, pointed space X.

Let Mf_p be the Thom spectrum of this map. Then:

Theorem 3.1 (Hopkins-Mahowald)**.** There is an equivalence of $\mathbb{E}_2$ ring spectra $Mf_p \simeq H\mathbb{F}_p$.

Since $\Omega^2 S^3 \simeq B\Omega^3 S^3$, we have that Mf_p is actually the homotopy orbits spectrum $(\mathbb{S}_p^\wedge)_{h\Omega^3 S^3}$. The remainder of this section will be devoted to proving this result.

3.3. **The proof.** Let's begin by showing that $\pi_0(Mf_p) \cong \mathbb{F}_p$. Then we'll use the 0th Postnikov stage to get our map $Mf_p \to H\mathbb{F}_p$.

Since

$$\pi_0 : \mathrm{Sp}^{\mathrm{cn}} \to \mathrm{Sp}^{\heartsuit} \simeq N(\mathrm{Ab})$$

as a functor on connective spectra is left adjoint to the Eilenberg-Maclane spectrum functor H, it preserves colimits. Hence we have that

$$\begin{aligned}
\pi_0(Mf_p) &= \pi_0((\mathbb{S}_p^\wedge)_{h\Omega^3 S^3}) \\
&= \pi_0\left(\operatorname*{colim}_{B\Omega^3 S^3}(\mathbb{S}_p^\wedge)\right) \\
&\cong \operatorname*{colim}_{B\Omega^3 S^3}(\pi_0(\mathbb{S}_p^\wedge)) \\
&\cong \operatorname*{colim}_{B\pi_0\Omega^3 S^3}(\pi_0(\mathbb{S}_p^\wedge)) \\
&\cong (\pi_0(\mathbb{S}_p^\wedge))_{\pi_0\Omega^3 S^3} \\
&\cong (\mathbb{Z}_p)_{\mathbb{Z}}
\end{aligned}$$

where the $\mathbb{Z}$ action on $\mathbb{Z}_p$ is given by multiplication by $1-p$ by construction. In particular, we can reduce our colimit to being a 1-colimit, because it is the colimit of the nerve NF of a functor $F : \pi_0\Omega^3 S^3 \to \mathrm{Ab}$ defined by taking the morphism 1 to multiplication by $1-p$, where $\pi_0\Omega^3 S^3$ is the one object category.

Then we have that

$$\pi_0(Mf_p) \cong (\mathbb{Z}_p)_{\mathbb{Z}} \cong \mathrm{Coeq}(\mathbb{Z}_p \underset{1}{\overset{1-p}{\rightrightarrows}} \mathbb{Z}_p) \cong \mathbb{Z}_p/(1-(1-p)) \cong \mathbb{F}_p.$$

Hence, we can now consider the map

$$\varphi : Mf_p \to H\mathbb{F}_p$$

down to the 0th stage of the Postnikov tower of Mf_p. Since the 0th truncation of connective spectra is symmetric monoidal, φ is an $\mathbb{E}_2$ map of ring spectra [Lur17, 2.2.1.10].

Since both Mf_p and $H\mathbb{F}_p$ are connective and p-complete—in particular, they are both p-torsion, as the unit of $\pi_*(Mf_p)$ is $1 \in \pi_0(Mf_p) \cong \mathbb{F}_p$—it suffices to show that φ is a p-adic equivalence, and thus, it suffices to show that

$$Mf_p \otimes H\mathbb{F}_p \simeq H\mathbb{F}_p \otimes H\mathbb{F}_p$$

i.e. that φ induces an isomorphism on $\mathbb{F}_p$-homology

$$\varphi_* : H_*(Mf_p; \mathbb{F}_p) \cong H_*(H\mathbb{F}_p; \mathbb{F}_p).$$

Now let us consider that we "change fiber" via the map $\mathbb{S}_p^\wedge \to H\mathbb{F}_p$ which is reduction mod p on homotopy groups, which induces a map $h : BGL_1(\mathbb{S}_p^\wedge) \to BGL_1(H\mathbb{F}_p)$. Then we have by our "change of fiber" formula that

$$M(h \circ f_p) \simeq Mf_p \otimes_{\mathbb{S}_p^\wedge} H\mathbb{F}_p \simeq Mf_p \otimes H\mathbb{F}_p,$$

where the last equivalence comes from the fact that Mf_p and $H\mathbb{F}_p$ are p-complete. In particular, by the definition of the relative tensor product in $\mathrm{Mod}_{\mathbb{S}_p^\wedge}$, we have that

$$\begin{aligned} Mf_p \otimes_{\mathbb{S}_p^\wedge} H\mathbb{F}_p &:= \mathrm{colim}(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} Mf_p \otimes \mathbb{S}_p^\wedge \otimes H\mathbb{F}_p \rightrightarrows Mf_p \otimes H\mathbb{F}_p) \\ &\simeq Mf_p \otimes H\mathbb{F}_p \end{aligned}$$

because e.g. $\mathbb{S}_p^\wedge \otimes H\mathbb{F}_p \simeq \mathbb{S} \otimes H\mathbb{F}_p \simeq H\mathbb{F}_p$, as $\mathbb{S}_p^\wedge$ is a p-completion and must be p-adically equivalent to $\mathbb{S}$.

Or even more directly, we have that $\mathbb{S}_p^\wedge \otimes \mathbb{S}/p \cong \mathbb{S}/p$ by definition as the Bousfield localization of $\mathbb{S}$ at $\mathbb{S}/p$, and then $\mathbb{S}/p \otimes H\mathbb{Z} \simeq H\mathbb{F}_p$, so

$$\mathbb{S}_p^\wedge \otimes H\mathbb{F}_p \simeq \mathbb{S}_p^\wedge \otimes \mathbb{S}/p \otimes H\mathbb{Z} \simeq \mathbb{S}/p \otimes H\mathbb{Z} \simeq H\mathbb{F}_p.$$

So, we have that $M(h \circ f_p) \simeq Mf_p \otimes H\mathbb{F}_p$, and furthermore, the composition $h \circ f_p$ is nullhomotopic, since $1 - p \equiv 1 \bmod p$; in other words,

$$\Omega^2 S^3 \xrightarrow{f_p} BGL_1(\mathbb{S}_p^\wedge) \xrightarrow{h} BGL_1(H\mathbb{F}_p)$$

is the trivial bundle, and so we have the $\mathbb{E}_2$-equivalence

$$Mf_p \otimes H\mathbb{F}_p \simeq \Sigma^\infty_+ \Omega^2 S^3 \otimes H\mathbb{F}_p.$$

Thus, showing that φ_* is an isomorphism is equivalent to showing that we have

$$\varphi_* : H_*(\Omega^2 S^3; \mathbb{F}_p) \xrightarrow{\cong} H_*(H\mathbb{F}_p; \mathbb{F}_p).$$

But since φ was constructed as an $\mathbb{E}_2$ map and hence preserves $\mathbb{E}_2$-Dyer-Lashof operations on $\mathbb{F}_p$-homology, it is again sufficient to show that φ_* is an isomorphism in degrees 0, 1. Since φ was constructed as the 0th Postnikov section of Mf_p, the isomorphism in degree 0 is automatic.

Then it suffices to show that we have a surjection in degree 1, since we know from the calculations that homology is $\mathbb{F}_p$ on both sides.

In particular, φ is automatically 1-connected, since it induces an isomorphism on π_0 by construction and a surjection in π_1 since $\pi_1(H\mathbb{F}_p) \cong 0$. But then connectivity of maps is defined by a fiber sequence, i.e. a map of spectra $X \to Y$ is 1-connected if and only if its fiber is a connective spectrum. Hence, $F \otimes H\mathbb{F}_p \to X \otimes H\mathbb{F}_p \to Y \otimes H\mathbb{F}_p$ also defines 1-connected map, because it is a cofiber sequence and thus is preserved by $- \otimes H\mathbb{F}_p$, and additionally, the fiber $F \otimes H\mathbb{F}_p$ is still connective because connective spectra are closed under products. Hence, the induced map

$$Mf_p \otimes H\mathbb{F}_p \to H\mathbb{F}_p \otimes H\mathbb{F}_p$$

is 1-connected, meaning that φ induces a surjection in degree 1 as desired.

4. A comparison of the proofs

The ideas of this discussion originate in [KN19, Appendix A.]. In some intuitive sense, both of the arguments we presented are already related in spirit by the fact that they reduce the result to a computation on degree 0, 1 $\mathbb{F}_p$-homology, since they both apply Steinberger's computations of Dyer-Lashof operations on the dual Steenrod algebra.

Whereas the algebraic argument directly characterizes the dual Steenrod algebra as a free $\mathbb{E}_2$-$H\mathbb{F}_p$-algebra, the topological argument first shows that $H\mathbb{F}_p$ (viewed as an $\mathbb{E}_2$-ring) arises as a Thom spectrum, and then uses its properties to show that it also gives the desired free algebra with $\mathbb{F}_p$ coefficients.

The connection between the two arises in [ACB], which characterizes Thom spectra of spheres as "weakly universal" algebras, i.e. so-called versal algebras. In particular, they define:

Definition 4.1. Let R be an $\mathbb{E}_{n+1}$ ring spectrum. Given $\chi : \Sigma^k R \to R$, the *versal R-algebra $R \mathbin{/\mkern-6mu/}_{\mathbb{E}_n} \chi$ of characteristic* χ is the following pushout in $\mathrm{Alg}_R^{\mathbb{E}_n}$:

$$\begin{tikzcd}
\mathrm{Free}_{\mathbb{E}_n}(\Sigma^k R) \arrow[r, "\tilde{0}"] \arrow[d, "\tilde{\chi}"'] \arrow[dr, phantom, "\ulcorner", very near end] & R \arrow[d] \\
R \arrow[r] & R \mathbin{/\mkern-6mu/}_{\mathbb{E}_n} \chi
\end{tikzcd}$$

where $\mathrm{Free}_{\mathbb{E}_n}$ denotes the free $\mathbb{E}_n$-algebra functor $\mathrm{Mod}_R \to \mathrm{Alg}_R^{\mathbb{E}_n}$ which is left adjoint to the forgetful functor, and the tildes denote adjoint maps.

A map $f : S^{k+1} \to BGL_1 R$ induces a map $\chi : \Sigma^k \mathbb{S} \otimes R \simeq \Sigma^k R \to R$ which we call its *corresponding characteristic*. For $k > 0$, it is given by the associated homotopy class $[f]$; for $k = 0$, we let $\chi = [f] - 1$. One can take the Thom spectrum of the corresponding n-fold loop map to obtain an $\mathbb{E}_n$-algebra, or one can take the versal algebra. In particular, it turns out that we have the following result [ACB, Theorem 4.10]:

Theorem 4.1. Given a map $S^{k+1} \to BGL_1 R$ with corresponding characteristic χ and corresponding n-fold loop map $f : \Omega^n \Sigma^n S^{k+1} \to BGL_1 R$, we have the following equivalence of $\mathbb{E}_n$ R-algebras:

$$Mf \simeq R \mathbin{/\mkern-6mu/}_{\mathbb{E}_n} \chi.$$

This is proven by demonstrating that both spectra corepresent the same functor on $\mathrm{Alg}_R^{\mathbb{E}_n}$: it turns out that for all $A \in \mathrm{Alg}_R^{\mathbb{E}_n}$ with characteristic χ, i.e. algebras such that composition with the unit $\Sigma^k R \to R \to A$ is nullhomotopic (the mapping spaces are empty otherwise),

$$\mathrm{Map}_{\mathrm{Alg}_R^{\mathbb{E}_n}}(Mf, A) \simeq \mathrm{Map}_{\mathrm{Alg}_R^{\mathbb{E}_n}}(R \mathbin{/\mkern-6mu/}_{\mathbb{E}_n} \chi, A) \simeq \Omega^{\infty+k+1} A,$$

which uses the fact that the $\mathbb{E}_n$ structure of Thom spectra has a universal characterization [ACB, Theorem 3.5].

In our case, for our map $f_p : \Omega^2 S^3 \to BGL_1(\mathbb{S}_p^\wedge)$, we have that

$$Mf_p \simeq \mathbb{S}_p^\wedge /\!\!/_{\mathbb{E}_2}\, p$$

where we denote the characteristic of f_p by p, since it is given by $[f_p] - 1 = (1-p) - 1 = -p$ which is equivalent to p in the pushout. Furthermore, the pushout and free construction both commute with $-\otimes H\mathbb{F}_p$, and since $\mathbb{S}_p^\wedge \otimes H\mathbb{F}_p \simeq H\mathbb{F}_p$, we obtain the pushout on the right:

$$\begin{tikzcd} \mathrm{Free}_{\mathbb{E}_2}(\mathbb{S}_p^\wedge) \arrow[r, "0"] \arrow[d, "p"'] & \mathbb{S}_p^\wedge \arrow[d] \\ \mathbb{S}_p^\wedge \arrow[r] & \mathbb{S}_p^\wedge /\!\!/_{\mathbb{E}_n}\, p \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathrm{Free}_{\mathbb{E}_2}(H\mathbb{F}_p) \arrow[r, "0"] \arrow[d, "0"'] & H\mathbb{F}_p \arrow[d] \\ H\mathbb{F}_p \arrow[r] & Mf_p \otimes H\mathbb{F}_p \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

Simultaneously, it is true that we have the pushout

$$\begin{tikzcd} \mathrm{Free}_{\mathbb{E}_2}(H\mathbb{F}_p) \arrow[r, "0"] \arrow[d, "0"'] & H\mathbb{F}_p \arrow[d] \\ H\mathbb{F}_p \arrow[r] & \mathrm{Free}_{\mathbb{E}_2}(\Sigma H\mathbb{F}_p) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

by definition of suspension as a pushout. Hence, showing $Mf_p \otimes H\mathbb{F}_p \simeq H\mathbb{F}_p \otimes H\mathbb{F}_p$ is exactly equivalent to showing that

$$\mathrm{Free}_{\mathbb{E}_2}(\Sigma H\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes H\mathbb{F}_p$$

where the left-hand side is exactly $\mathbb{F}_p[\Omega^2 S^3]$, i.e. the free $\mathbb{E}_2$ $H\mathbb{F}_p$-algebra on a generator in degree 1. We can see this explicitly using our discussion under Definition 1.1, the definition of $\mathrm{Free}_{\mathbb{E}_2}$ as an adjoint, and the fact that Σ is a pushout and thus commutes with Σ^∞:

$$\begin{aligned} \mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{H\mathbb{F}_p}}(\mathrm{Free}_{\mathbb{E}_2}(\Sigma H\mathbb{F}_p), A) &\simeq \mathrm{Map}_{\mathrm{Mod}_{H\mathbb{F}_p}}(\Sigma H\mathbb{F}_p, A) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{H\mathbb{F}_p}}(\Sigma \mathbb{S} \otimes H\mathbb{F}_p, A) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma\mathbb{S}, A) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma\Sigma^\infty_+ *, A) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty \Sigma(*_+), A) \\ &\simeq \mathrm{Map}_{\mathcal{S}_*}(S^1, \Omega^\infty A) \\ &\simeq \mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{H\mathbb{F}_p}}(\mathbb{F}_p[\Omega^2 S^3], A). \end{aligned}$$

Hence, the proof of Bökstedt using the Hopkins-Mahowald result is equivalent to the algebraic argument. In other words, since the Thom spectra of spheres are characterized by their multiplicativity in a “versal” way, this can sometimes allow us to simplify the “topological” arguments that use them into arguments that feel more “purely algebraic”.

References

[ABG$^+$14] Matthew Ando, Andrew J Blumberg, David Gepner, Michael J Hopkins, and Charles Rezk. An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology. *Journal of Topology*, 7(3):869–893, 2014.

[ACB] Omar Antolín-Camarena and Tobias Barthel. A simple universal property of Thom ring spectra, arxiv e-prints (2014). *arXiv preprint arXiv:1411.7988.*

[BCS10] Andrew J Blumberg, Ralph L Cohen, and Christian Schlichtkrull. Topological Hochschild homology of Thom spectra and the free loop space. *Geometry & Topology*, 14(2):1165–1242, 2010.

[BMMS86] Robert R Bruner, J Peter May, James E McClure, and Mark Steinberger. *H_∞ ring spectra and their applications*, volume 1176. Springer, 1986.

[DL62] Eldon Dyer and Richard K Lashof. Homology of iterated loop spaces. *American Journal of Mathematics*, 84(1):35–88, 1962.

[KA56] Tatsuji Kudo and Shôrô Araki. Topology of H_n-spaces and H-squaring operations. *Memoirs of the Faculty of Science, Kyushu University. Series A, Mathematics*, 10(2):85–120, 1956.

[KN18] Achim Krause and Thomas Nikolaus. Lectures on topological Hochschild homology and cyclotomic spectra. *preprint*, 2018.

[KN19] Achim Krause and Thomas Nikolaus. Bökstedt periodicity and quotients of DVRs, 2019.

[Lur17] Jacob Lurie. *Higher Algebra.* 2017.